数理解析研究所講究録 1404 巻 2004 年 32-44

Numeration systems, fractals and stochastic processes

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This is a trimmed version of [1], which can be downloaded from http://www.sci.osaka-cu.ac.jp/~kamae.

1 Numeration systems and colored tiling space

By a *numeration system*, we mean a compact metrizable space Ω with at least 2 elements as follows:

1. There exists a nontrivial closed multiplicative subgroup G of $\mathbb{R}_+$ such that $(\mathbb{R}, G)$ *acts numerically* to Ω in the sense that there exist continuous mappings $\chi_1 : \Omega \times \mathbb{R} \to \Omega$ and $\chi_2 : \Omega \times G \to \Omega$, where we denote $\omega + t := \chi_1(\omega, t)$, $\lambda\omega := \chi_2(\omega, \lambda)$, satisfying that

$$\omega + 0 = \omega,\ (\omega + t) + s = \omega + (t + s)$$
$$1\omega = \omega,\ \tau(\lambda\omega) = (\tau\lambda)\omega$$
$$\lambda(\omega + t) = \lambda\omega + \lambda t$$

for any $\omega \in \Omega$, $t, s \in \mathbb{R}$ and $\lambda, \tau \in G$.

2. The additive action of $\mathbb{R}$ to Ω is minimal and uniquely ergodic having 0-topological entropy.

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3. The multiplicative action of $\lambda(\in G)$ to Ω has $|\log\lambda|$-topological entropy. Moreover, the unique invariant probability measure under the additive action is invariant under the G-action and is the unique invariant probability measure attaining the topological entropy of the multiplication by $\lambda \neq 1$.

Note that if Ω is a numeration system, then Ω is a connected space with the continuum cardinality. Also, note that the multiplicative group G as above is either $\mathbb{R}_+$ or $\{\lambda^n;\ n \in \mathbb{Z}\}$ for some $\lambda > 1$. Moreover, the additive action is faithful, that is $\omega + t = \omega$ implies $t = 0$ for any $\omega \in \Omega$ and $t \in \mathbb{R}$.

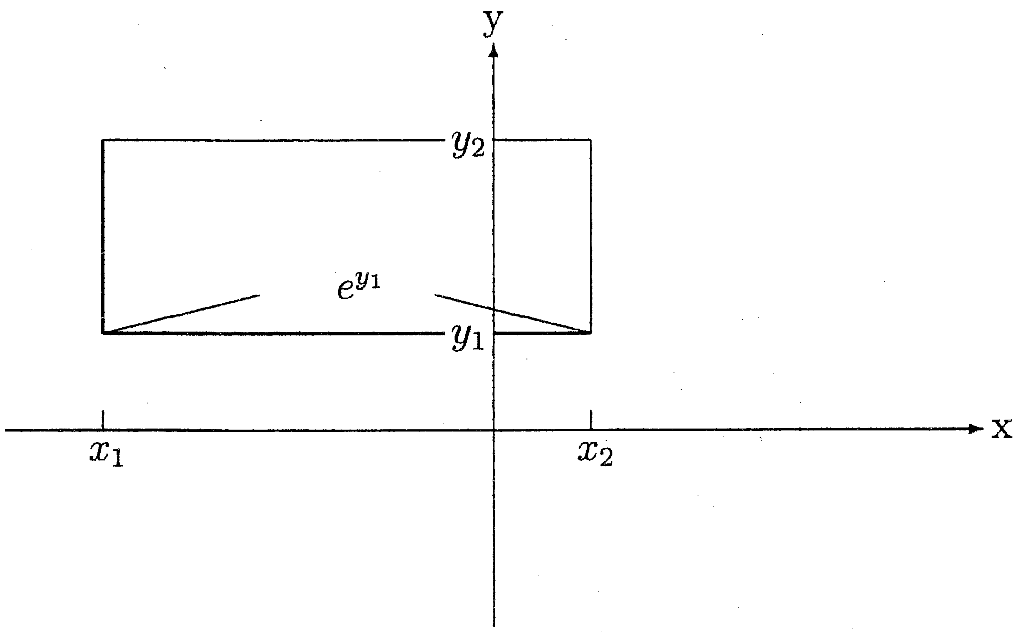

Figure 1: admissible tiles

Let $\mathbb{A}$ be a nonempty finite set. An element in $\mathbb{A}$ is called a *color*. A rectangle $[x_1, x_2)\times[y_1, y_2)$ in $\mathbb{R}^2$ is called an *admissible tile* if $x_2 - x_1 = e^{y_1}$ is satisfied (see Figure 1). A *colored tiling* ω with colors in $\mathbb{A}$ is a mapping from $dom(\omega)$ to $\mathbb{A}$, where $dom(\omega)$ consists of admissible tiles which are disjoint each other and the union of which is $\mathbb{R}^2$. For $R \in dom(\omega)$, $\omega(R)$ is considered as the color painted on the admissible tile R. In another word, a colored tiling is a partition of $\mathbb{R}^2$ by admissible tiles with colors in $\mathbb{A}$. Let $\Omega(\mathbb{A})$ be the set of colored tilings with colors in $\mathbb{A}$

A topology is introduced on $\Omega(\mathbb{A})$ so that a net $\{\omega_n\}_{n\in I} \subset \Omega(\mathbb{A})$ converges to $\omega \in \Omega(\mathbb{A})$ if for every $R \in dom(\omega)$, there exist $R_n \in dom(\omega_n)$ $(n \in I)$ such that

$$\omega(R) = \omega_n(R_n) \text{ for any sufficiently large } n \in I \text{ and } \lim_{n\to\infty} \rho(R, R_n) = 0,$$

where ρ is the Hausdorff metric.

For an admissible tile $R := [x_1, x_2) \times [y_1, y_2)$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we denote

$$\begin{aligned} R+t &:= [x_1 - t, x_2 - t) \times [y_1, y_2) \\ \lambda R &:= [\lambda x_1, \lambda x_2) \times [y_1 + \log\lambda, y_2 + \log\lambda). \end{aligned}$$

For $\omega \in \Omega(\mathbb{A})$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we define $\omega + t \in \Omega(\mathbb{A})$ and $\lambda\omega \in \Omega(\mathbb{A})$ as follows:

$$\begin{aligned} dom(\omega + t) &:= \{R + t;\ R \in dom(\omega)\} \\ (\omega + t)(R + t) &:= \omega(R) \text{ for any } R \in dom(\omega) \\ dom(\lambda\omega) &:= \{\lambda R;\ R \in dom(\omega)\} \\ (\lambda\omega)(\lambda R) &:= \omega(R) \text{ for any } R \in dom(\omega). \end{aligned}$$

Thus, $(\mathbb{R}, \mathbb{R}_+)$ acts numerically to $\Omega(\mathbb{A})$. We construct compact metrizable subspaces of $\Omega(\mathbb{A})$ corresponding to weighted substitutions which are numeration systems.

2 Weighted substitutions

A *substitution* σ on a set $\mathbb{A}$ is a mapping $\mathbb{A} \to \mathbb{A}^+$, where $\mathbb{A}^+ = \bigcup_{\ell=1}^{\infty} \mathbb{A}^\ell$. For $\xi \in \mathbb{A}^+$, we denote $|\xi| := \ell$ if $\xi \in \mathbb{A}^\ell$, and ξ with $|\xi| = \ell$ is usually denoted by $\xi_0\xi_1\cdots\xi_{\ell-1}$ with $\xi_i \in \mathbb{A}$. We can extend σ to be a homomorphism $\mathbb{A}^+ \to \mathbb{A}^+$ as follows:

$$\sigma(\xi) := \sigma(\xi_0)\sigma(\xi_1)\cdots\sigma(\xi_{\ell-1}),$$

where $\xi \in \mathbb{A}^\ell$ and the right-hand side is the concatenations of $\sigma(\xi_i)$'s. We can define $\sigma^2, \sigma^3, \cdots$ as the compositions of $\sigma : \mathbb{A}^+ \to \mathbb{A}^+$.

A *weighted substitution* (σ, τ) on $\mathbb{A}$ is a mapping $\mathbb{A} \to \mathbb{A}^+ \times (0,1)^+$ such that $|\sigma(a)| = |\tau(a)|$ and $\sum_{i<|\tau(a)|} \tau(a)_i = 1$ for any $a \in \mathbb{A}$. Note that σ is a substitution on $\mathbb{A}$. We define $\tau^n : \mathbb{A} \to (0,1)^+$ $(n = 2, 3, ...)$ inductively by

$$\tau^n(a)_k = \tau(a)_i \tau^{n-1}(\sigma(a)_i)_j$$

for any $a \in \mathbb{A}$ and i, j, k with

$$0 \le i < |\sigma(a)|,\ 0 \le j < |\sigma^{n-1}(\sigma(a)_i)|,\ k = \sum_{h<i} |\sigma^{n-1}(\sigma(a)_h)| + j.$$

Then, (σ^n, τ^n) is also a weighted substitution for $n = 2, 3, \cdots$.

A substitution σ on $\mathbb{A}$ is called *primitive* if there exists a positive integer n such that for any $a, a' \in \mathbb{A}$, $\sigma^n(a)_i = a'$ holds for some i with $0 \le i < |\sigma^n(a)|$.

For a weighted substitution (σ, τ) on $\mathbb{A}$, we always assume that

$$\textit{the substitution } \sigma \textit{ is primitive.} \tag{1}$$

We define the *base set* $B(\sigma, \tau)$ as the closed, multiplicative subgroup of $\mathbb{R}_+$ generated by the set

$$\left\{ \begin{array}{ll} \tau^n(a)_i\ ; & a \in \mathbb{A},\ n = 0, 1, \cdots \text{ and } 0 \le i < |\sigma^n(a)| \\ & \text{such that } \sigma^n(a)_i = a \end{array} \right\}.$$

Let $G := B(\sigma, \tau)$. Then, there exists a function $g : \mathbb{A} \to \mathbb{R}_+$ such that

$$g(\sigma(a)_i)G = g(a)\tau(a)_i G \tag{2}$$

for any $a \in \mathbb{A}$ and $0 \le i < |\sigma(a)|$. Note that if $G = \mathbb{R}_+$, then we can take $g \equiv 1$. In the other case, we can define g by $g(a_0) = 1$ and $g(a) := \tau^n(a_0)_i$ for some n and i such that $\sigma^n(a_0)_i = a$, where a_0 is any fixed element in $\mathbb{A}$.

Let (σ, τ) be a weighted substitution satisfying (1). Let $G = B(\sigma, \tau)$. Let g satisfy (2). Let $\Omega(\sigma, \tau, g)'$ be the set of all elements ω in $\Omega(\mathbb{A})$ such that for any $[x_1, x_2) \times [y_1, y_2) \in dom(\omega)$ with $\omega([x_1, x_2) \times [y_1, y_2)) = a$, we have

(I) $e^{y_1} \in g(a)G$, and

(II) $R^i \in dom(\omega)$ and $\omega(R^i) = \sigma(a)_i$ hold for $i = 0, 1, \cdots, |\sigma(a)| - 1$, where

$$\begin{aligned} R^i &:= [x_1 + (x_2 - x_1)\sum_{j=0}^{i-1} \tau(a)_j,\ x_1 + (x_2 - x_1)\sum_{j=0}^{i} \tau(a)_j) \\ &\quad \times [y_1 + \log \tau(a)_i,\ y_1). \end{aligned}$$

A vertical line $\gamma := \{x\} \times (-\infty, \infty)$ is called a *separating line* of $\omega \in \Omega(\sigma, \tau, g)'$ if for any $R \in dom(\omega)$, $R^\circ \cap \gamma = \emptyset$, where R° denotes the set of inner points of R. Let $\Omega(\sigma, \tau, g)''$ be the set of all $\omega \in \Omega(\sigma, \tau, g)'$ which do not have a separating line and $\Omega(\sigma, \tau, g)$ be the closure of $\Omega(\sigma, \tau, g)''$. Then, $(\mathbb{R}, G)$ acts to $\Omega(\sigma, \tau, g)$ numerically. We usually denote $\Omega(\sigma, \tau, 1)$ simply by $\Omega(\sigma, \tau)$.

Theorem 1. *The space $\Omega(\sigma, \tau, g)$ is a numeration system with $G = B(\sigma, \tau)$.*

Theorem 2. *Let Ω be a numeration system with $G = \mathbb{R}_+$, that is, with the multiplicative $\mathbb{R}_+$-action. Then, the additive action on the probability space Ω with the unique invariant probability measure μ has a pure Lebesgue spectrum.*

3 ζ-function

Let $\Omega := \Omega(\sigma, \tau, g)$ satisfying (1) and (2). For $\alpha \in \mathbb{C}$, we define the associated matrices on the suffix set $\mathbb{A} \times \mathbb{A}$ as follows:

$$\begin{aligned} M_\alpha &:= \left(\sum_{i;\sigma(a)_i = a'} \tau(a)_i^\alpha \right)_{a,a' \in \mathbb{A}} \qquad (3) \\ M_{\alpha,+} &:= \left(1_{\sigma(a)_0 = a'}\ \tau(a)_0^\alpha \right)_{a,a' \in \mathbb{A}} \\ M_{\alpha,-} &:= \left(1_{\sigma(a)_{|\sigma(a)|-1} = a'}\ \tau(a)_{|\sigma(a)|-1}^\alpha \right)_{a,a' \in \mathbb{A}} \end{aligned}$$

Let Θ be the set of *closed orbits* of Ω with respect to the action of G. That is, Θ is the family of subsets ξ of Ω such that $\xi = G\omega$ for some $\omega \in \Omega$ with $\lambda\omega = \omega$ for some $\lambda \in G$ with $\lambda > 1$. We call λ as above a *multiplicative cycle* of ξ. The minimum multiplicative cycle of ξ is denoted by $c(\xi)$.

We say that $\xi \in \Theta$ has a *separating line* if $\omega \in \xi$ has a separating line. Note that in this case, the separating line is necessarily the y-axis and is in common among $\omega \in \xi$. Denote by Θ_0 the set of $\xi \in \Theta$ with the separating line.

Define the ζ-function of G-action to Ω by

$$\zeta_\Omega(\alpha) := \prod_{\xi\in\Theta}(1 - c(\xi)^{-\alpha})^{-1}, \tag{4}$$

where the infinite product converges for any $\alpha \in \mathbb{C}$ with $\mathcal{R}(\alpha) > 1$. It is extended to the whole complex plane by the analytic extension.

Theorem 3. *We have*

$$\zeta_\Omega(\alpha) = \frac{\det(I - M_{\alpha,+})\det(I - M_{\alpha,-})}{\det(I - M_\alpha)}\,\zeta_{\Sigma_0}(\alpha),$$

where

$$\zeta_{\Sigma_0}(\alpha) := \prod_{\xi\in\Theta_0}(1 - c(\xi)^{-\alpha})^{-1}$$

is a finite product with respect to $\xi \in \Theta_0$.

Theorem 4. *(i)* $\zeta_\Omega(\alpha) \neq 0$ *if* $\mathcal{R}(\alpha) \neq 0$.

(ii) In the region $\mathcal{R}(\alpha) \neq 0$, α *is a pole of* $\zeta_\Omega(\alpha)$ *with multiplicity* k *if and only if it is a of zero of* $\det(I - M_\alpha)$ *with multiplicity* k *for any* $k = 1, 2, \cdots$.

(iii) 1 *is a simple pole of* $\zeta_\Omega(\alpha)$.

Theorem 5. *For* $\Omega = \Omega(\sigma, \eta, g)$, *if* $B(\sigma,\tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$ *with* $\lambda > 1$, *then there exist polynomials* $p, q \in \mathbb{Z}[z]$ *such that* $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$. *Conversely, if* $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$ *holds for some polynomials* $p, q \in \mathbb{Z}[z]$ *and* $\lambda > 1$, *then* $B(\sigma,\tau) = \{\lambda^{kn};\ n \in \mathbb{Z}\}$ *for some positive integer* k.

Theorem 6. *If* $B(\sigma,\tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$, *then* λ *is an algebraic number.*

4 β-expansion system

Let β be an algebraic integer with $\beta > 1$ such that 1 has the following periodic β-expansion

$$
\begin{aligned}
&1 = (b_1 0^{i_1-1} b_2 0^{i_2-1} \cdots b_k 0^{i_k-1})^\infty \\
&b_1, b_2, \cdots, b_k \in \{1, 2, \cdots, \lfloor \beta \rfloor\} \\
&i_1, i_2, \cdots, i_k \in \{1, 2, \cdots\},
\end{aligned}
$$

where $(\ \ )^\infty$ implies the infinite time repetition of $(\ \ )$. Let $n := i_1 + i_2 + \cdots + i_k \geq 1$ and assume that n is the minimum period of the above sequence. Since the above sequence is the expansion of 1, we have the solution of the following equation in $a_1, a_2, \cdots, a_{k+1}$ with $a_1 = a_{k+1} = 1$ and $0 < a_j < 1$ $(j = 2, \cdots, k)$:

$$a_j = b_j \beta^{-1} + a_{j+1} \beta^{-i_j} \ (j = 1, 2, \cdots, k).$$

Let $\mathbb{A} := \{1, 2, \cdots, k\}$ and define a weighted substitution (σ, τ) by

$$
\begin{gathered}
j \to (1,\ (1/a_j)\beta^{-1})^{b_j}\ (j+1,\ (a_{j+1}/a_j)\beta^{-i_j}) \\
(j = 1, 2, \cdots, k-1) \\
k \to (1,\ (1/a_k)\beta^{-1})^{b_k}\ (1,\ (a_{k+1}/a_k)\beta^{-i_k})
\end{gathered}
$$

where $(\ ,\ )^k$ implies the k-time repetition of $(\ ,\ )$. Then, σ is primitive and $B(\sigma, \tau) = \{\beta^n;\ n \in \mathbb{Z}\}$. Define $g : \mathbb{A} \to \mathbb{R}_+$ by $g(j) := a_j$. Then, g satisfies (2) and $\Omega(\sigma, \tau, g)$ is a numeration system by Theorem 2. We denote $\Omega(\beta) := \Omega(\sigma, \tau, g)$ and $\Omega(\beta)$ is called the *β-expansion system.*

Theorem 7. *We have*

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1 - \beta^{-\alpha}}{1 - \sum_{j=1}^{k} b_j \beta^{-(i_1 + \cdots + i_{j-1} + 1)\alpha} - \beta^{-n\alpha}}.$$

5 homogeneous cocycles and fractals

Let $\Omega := \Omega(\sigma, \tau, g)$ satisfy (1) and (2).

A continuous function $F : \Omega \times \mathbb{R} \to \mathbb{C}$ is called a *cocycle* on Ω if

$$F(\omega, t+s) = F(\omega, t) + F(\omega + t, s) \tag{5}$$

holds for any $\omega \in \Omega$ and $s, t \in \mathbb{R}$. A cocycle F on Ω is called α-*homogeneous* if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega, t)$$

for any $\omega \in \Omega$, $\lambda \in G$ and $t \in \mathbb{R}$, where α is a given complex number. A cocycle $F(\omega, t)$ on Ω is called *adapted* if there exists a function $\Xi : \mathbb{A} \times \mathbb{R}_+ \to \mathbb{C}$ such that

$$F(\omega, x_2) - F(\omega, x_1) = \Xi(\omega(R), x_2 - x_1) \tag{6}$$

for any $\omega \in \Omega$ and tile $R := [x_1, x_2) \times [y_1, y_2) \in dom(\omega)$.

In [2], nonzero adapted α-homogeneous cocycles on Ω with $0 < \alpha < 1$ is characterized. In fact, we have

Theorem 8. *A nonzero adapted α-homogeneous cocycle on Ω is characterized by (6) with α and Ξ satisfying that $\mathcal{R}(\alpha) > 0$ and there exists a nonzero vector $\xi = (\xi_a)_{a \in \mathbb{A}}$ such that $M_\alpha \xi = \xi$ (see (3)) and $\Xi(\omega(R), x_2 - x_1) = (x_2 - x_1)^\alpha \xi_{\omega(R)}$ for any tile $R := [x_1, x_2) \times [y_1, y_2) \in dom(\omega)$. Hence, a nonzero adapted α-homogeneous cocycle exists if and only if $\mathcal{R}(\alpha) > 0$ and α is a pole of $\zeta_\Omega(\alpha)$.*

It is known [2] that

Theorem 9. *Let μ be the unique invariant probability measure on Ω under the additive action. Let $0 < \alpha < 1$. For a nonzero α-homogeneous cocycle F on Ω, we have the following results.*

(i) There exists a constant C such that

$$|F(\omega, t) - F(\omega, s)| \leq C|t - s|^\alpha$$

for any $\omega \in \Omega$ and $s, t \in \mathbb{R}$. That is, the functions $F(\omega, t)$ on t for $\omega \in \Omega$ are uniformly α-Hölder continuous.

(ii) For any $\omega \in \Omega$ and $t \in \mathbb{R}$,

$$\limsup_{s \downarrow 0} \frac{1}{s^\alpha} |F(\omega, t+s) - F(\omega, t)| > 0$$

holds. That is, for any $\omega \in \Omega$ the function $F(\omega, \cdot)$ is nowhere locally α'-Hölder continuous for any $\alpha' > \alpha$. In particular, $F(\omega, \cdot)$ is nowhere differentiable.

(iii) The stochastic process $F(\omega, t)$ with time parameter $t \in \mathbb{R}$ and random element $\omega \in \Omega$ with respect to μ has a strictly ergodic stationary increment having 0-entropy.

(iv) $F(\omega, \lambda t)$ has the same law as $\lambda^\alpha F(\omega, t)$ for any $\lambda \in G$. Hence, the process $F(\omega, t)$ is α-self similar if $G = \mathbb{R}_+$.

(v) $\int F(\omega, t) d\mu(\omega) = 0$ for any $t \in \mathbb{R}$.

Example 1. Let $\mathbb{A} = \{+, -\}$ and (σ, τ) be a weighted substitution such that

$$\begin{aligned} + &\to (+, 4/9)(-, 1/9)(+, 4/9) \\ - &\to (-, 4/9)(+, 1/9)(-, 4/9). \end{aligned}$$

Then, $4/9 \in B(\sigma, \tau)$ since $\sigma(+)_0 = +$ and $\tau(+)_0 = 4/9$. Moreover, $1/81 \in B(\sigma, \tau)$ since $\sigma^2(+)_4 = +$ and $\tau^2(+)_4 = 1/81$. Since $4/9$ and $1/81$ do not have a common multiplicative base, we have $B(\sigma, \tau) = \mathbb{R}_+$. Let $\Omega = \Omega(\sigma, \tau)$. Then we have

$$\zeta_\Omega(\alpha) = \frac{1}{(1 - 2(4/9)^\alpha - (1/9)^\alpha)(1 - 2(4/9)^\alpha + (1/9)^\alpha)},$$

so that $1/2$ is a simple pole of ζ_Ω. In fact, the associated matrix

$$M_{1/2} = \begin{pmatrix} 4/3 & 1/3 \\ 1/3 & 4/3 \end{pmatrix}$$

has an eigen-vector $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigen-value 1. Let F be the $1/2$-homogeneous adapted cocycle on Ω defined by the equation:

$$F(\omega, x_2) - F(\omega, x_1) = \pm(x_2 - x_1)^{1/2}$$

if there exists a tile $[x_1, x_2) \times [y_1, y_2)$ in ω with color $\pm$, respectively (see Theorem 9).

Then, $F(\omega, t)$ is a $1/2$-selfsimilar process with respect to the unique invariant measure μ under the additive action, called N-process ([3]).

Let $\mathcal{I}(\Omega)$ be the set of $\omega \in \Omega$ such that there exists $[x_1, x_2) \times [y_1, y_2) \in dom(\omega)$ satisfying that $x_1 = 0$ and $y_1 \leq 0 < y_2$. An element $\omega \in \mathcal{I}(\Omega)$ is called an *integer* in Ω. Let

$$\mathcal{II}(\Omega) := \{(\omega, t) \in \mathcal{I}(\Omega) \times \mathbb{R};\ \omega + t \in \mathcal{I}(\Omega)\}.$$

A continuous function $F : \mathcal{II}(\Omega) \to \mathbb{C}$ is called a cocycle on $\mathcal{I}(\Omega)$ if (5) is satisfied for any $\omega \in \mathcal{I}(\Omega)$ and $t, s \in \mathbb{R}$ such that $(\omega, t) \in \mathcal{II}(\Omega)$ and $(\omega, t+s) \in \mathcal{II}(\Omega)$.

A cocycle F on $\mathcal{I}(\Omega)$ is called adapted if there exists a function $\Xi : \mathbb{A} \times \mathbb{R}_+ \to \mathbb{C}$ such that (6) is satisfied for any $\omega \in \mathcal{I}(\Omega)$ and tile $[x_1, x_2) \times [y_1, y_2) \in dom(\omega)$ with $y_2 > 0$. Let $\alpha \in \mathbb{C}$. A cocycle F on $\mathcal{I}(\Omega)$ is called α-homogeneous if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega, t)$$

for any $(\omega, t) \in \mathcal{II}(\Omega)$ and $\lambda \in G$ with $(\lambda\omega, \lambda t) \in \mathcal{II}(\Omega)$. Note that if $(\omega, t) \in \mathcal{II}(\Omega)$, then for any $\lambda \in G$ with $\lambda > 1$, $(\lambda\omega, \lambda t) \in \mathcal{II}(\Omega)$ holds.

A cocycle F on $\mathcal{I}(\Omega)$ is called a *coboundary* on $\mathcal{I}(\Omega)$ if there exists a continuous function $G : \mathcal{I}(\Omega) \to \mathbb{R}^k$ such that

$$F(\omega, t) = G(\omega + t) - G(\omega)$$

for any $(\omega, t) \in \mathcal{II}(\Omega)$.

The following theorem is proved in [4].

Theorem 10. *A nonzero adapted α-homogeneous cocycle on $\mathcal{I}(\Omega)$ with $\mathcal{R}(\alpha) < 0$ is characterized by (6) with Ξ satisfying that there exists a nonzero vector $\xi = (\xi_a)_{a\in\mathbb{A}}$ such that $M_\alpha \xi = \xi$ (see (3)) and $\Xi(\omega(R), x_2 - x_1) = (x_2 - x_1)^\alpha \xi_{\omega(R)}$ for any tile $R := [x_1, x_2) \times [y_1, y_2) \in dom(\omega)$ with $y_2 > 0$. Hence, a nonzero adapted α-homogeneous cocycle on $\mathcal{I}(\Omega)$ with $\mathcal{R}(\alpha) < 0$ exists if and only if α is a pole of $\zeta_\Omega(\alpha)$. Moreover, any cocycle as this is a coboundary.*

Example 2. Let us consider the β-expansion system with $\beta > 1$ such that $\beta^3 - \beta^2 - \beta - 1 = 0$. Then the expansion of 1 is $(110)^\infty$

and the corresponding weighted substitution is

$$\begin{aligned} 1 &\rightarrow (1,\ \beta^{-1})(2,\ \beta^{-2}+\beta^{-3}) \\ 2 &\rightarrow (1,\ \frac{\beta}{\beta+1})(1,\ \frac{1}{\beta+1}) \end{aligned}$$

Denote $\Omega := \Omega(\beta)$. The associated matrix is

$$M_\alpha = \begin{pmatrix} \beta^{-\alpha} & (\beta^{-2}+\beta^{-3})^\alpha \\ \frac{\beta^\alpha+1}{(\beta+1)^\alpha} & 0 \end{pmatrix}$$

Let γ be one of the complex solutions of the equation $z^3 - z^2 - z - 1 = 0$. Then, $|\gamma| < 1$. Let $\alpha \in \mathbb{C}$ be such that $\gamma = \beta^\alpha$. Then, $\mathcal{R}(\alpha) < 0$. Since we have

$$M_\alpha \begin{pmatrix} 1 \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ \delta \end{pmatrix}$$

with $\delta := \dfrac{\beta^\alpha + 1}{(\beta+1)^\alpha}$, there exists an α-homogeneous adapted cocycle F on $\mathcal{I}(\Omega)$ satisfying that

$$F(\omega, x_2) - F(\omega, x_1) = \begin{cases} (x_2 - x_1)^\alpha & (\omega(R) = 1) \\ \delta(x_2 - x_1)^\alpha & (\omega(R) = 2) \end{cases}$$

if there exists $R := [x_1, x_2) \times [y_1, y_2) \in dom(\omega)$ with $y_2 > 0$.

For $\omega \in \mathcal{I}(\Omega)$, let $R_0(\omega)$ be the tile $[x_0, x_1) \times [y_0, y_1) \in \omega$ such that $x_0 = 0$ and $y_0 \leq 0 < y_1$. For $i = 0, 1, 2, \cdots$, let R_i be the i-th ancestor of $R_0(\omega)$. Let $Corner(R_i) =: (x_i, y_i)$. Let

$$G(\omega) := \sum_{i=0}^{\infty} (x_i - x_{i+1})^\alpha.$$

Since if $x_i > x_{i+1}$, then there exists a tile $[x_{i+1}, x_i) \times [y_{i+1}, y_{i+1} + \log\beta)$ with color 1 in ω, we have

$$F(\omega, x_i) - F(\omega, x_{i+1}) = (x_i - x_{i+1})^\alpha$$

for any $i = 0, 1, \cdots$.

Take any $t \in \mathbb{R}$ such that $(\omega, t) \in \mathcal{II}(\Omega)$. Let $(R'_i)_{i=1,2,\cdots}$ and $(x'_i)_{i=0,1,\cdots}$ be the sequences as above for $\omega + t$ instead of ω. Then, there exist $i_0 \geq 1$, $j_0 \geq 1$ such that $R'_{i_0+k} = R_{j_0+k} + t$ for any $k = 0, 1, \cdots$. Then, since $x'_{j_0+k} = x_{i_0+k} - t$ for any $k = 0, 1, \cdots$, we have

$$
\begin{aligned}
& G(\omega + t) - G(\omega) \\
= & \sum_{i=0}^{j_0-1} (x'_i - x'_{i+1})^\alpha - \sum_{i=0}^{i_0-1} (x_i - x_{i+1})^\alpha \\
= & -F(\omega + t, x'_{j_0}) + F(\omega, x_{i_0}) = F(\omega, t).
\end{aligned}
$$

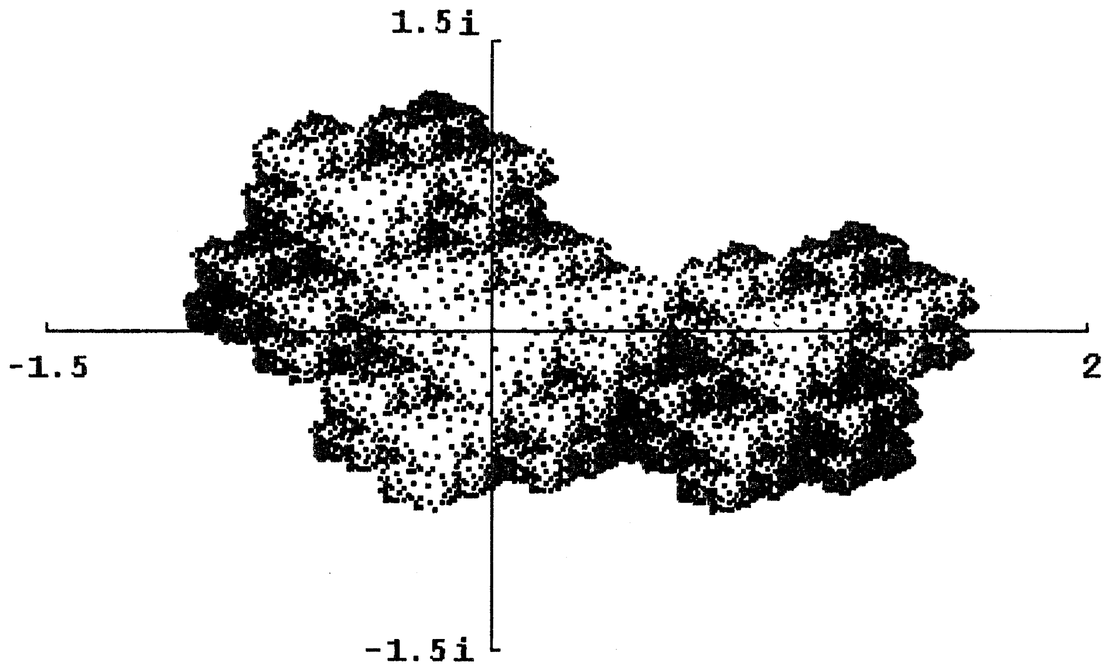

Figure 2: $G(\mathcal{I}(\Omega))$

Thus, the α-homogeneous cocycle F is a coboundary with coboundary function G. The set $G(\mathcal{I}(\Omega))$ is known as Rauzy fractal which is shown in Figure 2.

6 Open problems

(1) Does a numeration system which is not homomorphic to any numeration system coming from weighted substitutions exists? If yes, how to characterize the numeration systems coming from weighted substitutions?
(2) Does the condition $B(\sigma, \tau) \neq \mathbb{R}_+$ imply that the $\mathbb{R}$-action of a numeration system coming from weighted substitutions with respect to the unique invariant probability measure is not weakly mixing? When doess it have the discrete spectrum?
(3) What is the multiplicity of the pure Lebesgue spectrum possessed by the $\mathbb{R}$-action of a numeration system coming from a weighted substitution with $B(\sigma, \tau) = \mathbb{R}_+$ with respect to the unique invariant probability measure?
(4) When does a numeration system admit an additive group structure consistent with the $(\mathbb{R}, G)$-action?

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